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UNSTEADY LIFT, MOMENT, AND CAVITY LENGTH CALCULATIONS FOR AN OSCILLATING PARTIALLY CAVITATED HYDROFOIL

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NOMENCLATURE

Φ	velocity potential of flow
ϕ	acceleration potential
U	free stream velocity
t	time
σ	normalized cavity pressure
d	fluid density
$M(t)$	slope of foil
$B(t)$	y intercept of foil
ℓ	semichord of foil
c	position of rear end of cavity
α	$\sqrt{\frac{\ell-c}{\ell+c}}$
A	$\sqrt{\frac{\sqrt{\alpha^2+1}+\alpha}{2}}$
B	$\sqrt{\frac{\sqrt{\alpha^2+1}-\alpha}{2}}$
f	force on foil
m	moment of foil
ω	cavity length/chord length $(\frac{\ell+c}{2\ell})$
f_o, m_o, ω_o	steady parts of f, m, ω
γ	frequency of oscillation of foil
Ω	reduced frequency $(\frac{\gamma\ell}{U})$
K_o, K_1	Bessel functions (see [3] page 172)

Γ	Theodorsen's function
δ_k	$\phi_k^+ - \phi_k^-$
δ	net normalized unsteady pressure on foil
f^*	normalized unsteady force
m^*	normalized unsteady moment
ω^*	normalized unsteady cavity length
f_s^*, f_c^*	real and imaginary parts of f^*
m_s^*, m_c^*	real and imaginary parts of m^*
ω_s^*, ω_c^*	real and imaginary parts of ω^*
M_o	steady part of M(t)
M_1, B_1	amplitudes of unsteady parts of M(t), B(t)

The following quantities have complex definitions given in [1]

C_k, ϕ_k	pages 4 and 7
$\psi_k, \tilde{\psi}_k$	Appendix VI
Q_1, Q_2	page 30
v_c, v_s	page 34
$d_{k,c}^*$, $d_{k,s}^*$	page 32 (see also page 22)
α_c, α_s	page 22
$J_{k,c}$, $J_{k,s}$	page 17
$I_{k,c}$, $I_{k,s}$	page 22
ϕ_α, $\tilde{\psi}_\alpha$, $J_{\alpha,c}$, $J_{\alpha,s}$, $I_{\alpha,c}$, $I_{\alpha,s}$	page 31

1. SUMMARY

This report contains the results of calculations of the unsteady force, moment, and cavity length of a partially cavitated hydrofoil subject to unit heave or pitch oscillations. These can be combined with arbitrary amplitude and phase. Included also are analytic descriptions of the zero-cavity-length limit and of the zero-frequency limit. Numerical results for these limiting cases are also included.

2. INTRODUCTION

A previous report[1] presents the theory for the unsteady motion of a partially cavitated hydrofoil. In order to carry out a hydroelastic analysis and also to indicate directions for further experimental and theoretical work, numerical calculations had to be made. The results of these calculations form the main body of this report.

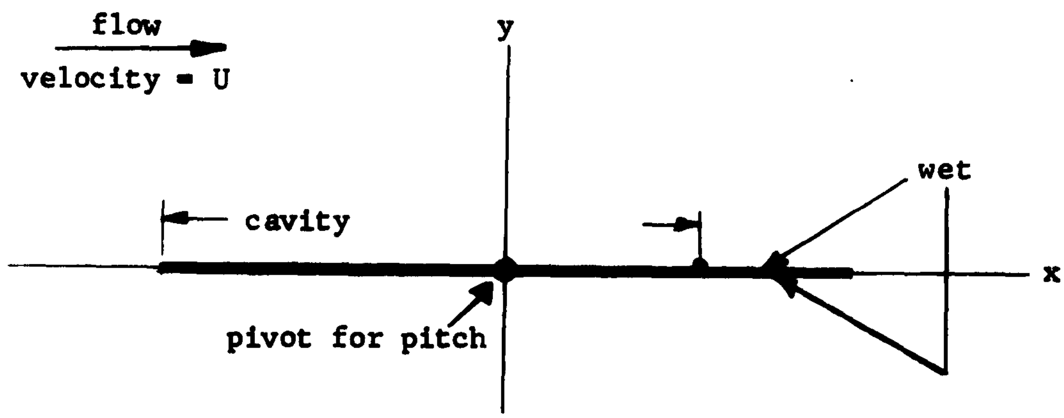
For certain limiting cases, namely the cavity length approaching zero and the reduced frequency approaching zero, we obtain simple expressions for the solutions. For the cavity length going to zero, we show that the limiting result agrees with Theodorsen's standard result[2].

The limiting result for zero reduced frequency was obtained by a quasi-steady analysis of the steady state solution. The results agree graphically with the limit of the results of the above-mentioned calculations.

A short summary of [1] is included. This makes it possible to use the results presented herein without reading all of [1].

3. SUMMARY OF TRG-153-SR-1

In [1] the problem under study was the determination of the flow around a partially cavitated hydrofoil. The foil was assumed to be a flat plate of infinite span and immersed in an infinite fluid. The free stream velocity and the cavitation number were assumed constant. The foil was assumed to be subject to unsteady motion, specifically, simple harmonic heaving and/or pitching, and the particular question raised was the determination of the unsteady terms for the flow in these cases.



Timman's general solution in terms of the acceleration potential ϕ, defined by $\phi = \Phi_t + U\Phi_x$ where Φ is the velocity potential, is

$$\phi = \sigma + \sum_{k=1}^{4} c_k(t)\phi_k$$

where ϕ_k are explicitly given* (Appendix VI), c_3 and c_4 are defined by the motion of the foil (page 7), while c_1 and c_2 are (along with the length of the cavity) to be determined. The four possible boundary conditions to determine the three unknown functions are stated, (page 6). The vertical component of the velocity $\bar{\phi}_y$, is derived in terms of ψ_k, the conjugate functions of ϕ_k, to be:

$$\bar{\phi}_y = \frac{1}{U}\left\{\sum_{k=1}^{4} c_k(t)\psi_k - \frac{1}{U}\frac{d}{dt}\int_{-\infty}^{x} c_k(t-\frac{x-x'}{U})\psi_k(x',y,\alpha(t-\frac{x-x'}{U})dx')\right\}$$

The steady state solution is then derived (pages 9-12) with the result

$$c_1 = \frac{U^2MB}{\alpha}(\alpha^2+1)(\alpha+\sqrt{\alpha^2+1})$$

$$c_2 = -\frac{U^2MB}{\alpha}(3\alpha^2 + 1 + 3\alpha\sqrt{\alpha^2+1})$$

with α (defined by $\alpha = \sqrt{\frac{\ell-c}{\ell+c}}$) given by:

$$\sigma = \frac{U^2M}{\alpha}(\sqrt{\alpha^2+1} + \alpha)^2 .$$

A second linearization is carried out and the problem is reduced to solving six linear equations in six unknowns (pages 13-24), where the six equations are given on page 23. The unsteady lift and moment are derived in terms of the unknown

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* All Appendix and page number references in this section are to [1].

constants (pages 25-28) and are given on page 28, along with the steady lift and moment.

In pages 29-35 we rewrite the equations in a non-dimensional form suitable for machine computation. We define solutions for unit heave and unit pitch as functions of ω_0 = cavity length/chord length and Ω = reduced frequency, and show how to combine these solutions for arbitrary pitch and heave.

4. ZERO CAVITY-LENGTH LIMIT

For analytical purposes it is convenient to represent the set of six real linear equations by three complex linear equations. Let $Q^* = Q_2 + iQ_1$, $v^* = v_s + iv_c$, and let $q^* = q_U + iq_L$ where

$$q_U = -\sum_3^4 \begin{pmatrix} d^*_{k,s}\phi_k(\infty) \\ d^*_{k,c}\left(J_{k,c} + \frac{(-1)^k}{\Omega}\right) + d^*_{k,s}\left(J_{k,s} + \frac{(3-k)}{\Omega^2} - \tilde{\psi}_k(\infty)\right) \\ d^*_{k,c}I_{k,c} + d^*_{k,s}I_{k,s} \end{pmatrix}$$

$$q_L = -\sum_3^4 \begin{pmatrix} d^*_{k,c}\phi_k(\infty) \\ d^*_{k,c}\left(J_{k,s} + \frac{3-k}{\Omega^2} - \tilde{\psi}_k(\infty)\right) - d^*_{k,s}\left(J_{k,c} + \frac{(-1)^k}{\Omega}\right) \\ d^*_{k,c}I_{k,s} - d^*_{k,s}I_{k,c} \end{pmatrix}$$

Then

$$q^* = -\sum_3^4 d^*_k \begin{pmatrix} \phi_k(\infty) \\ M^*_k \\ I^*_k \end{pmatrix}$$

where

$$d^*_k = d^*_{k,s} + id^*_{k,c}$$

$$M^*_k = J_{k,s} + \frac{3-k}{\Omega^2} - \tilde{\psi}_k(\infty) - i\left(J_{k,c} + \frac{(-1)^k}{\Omega}\right)$$

$$= -i\left(J_{k,c} + \frac{(-1)^k}{\Omega} + i\left(J_{k,s} + \frac{3-k}{\Omega^2} - \tilde{\psi}_k(\infty)\right)\right) = -iN^*_k$$

$$I^*_k = -i(I_{k,c} + iI_{k,s}) = -iL^*_k$$

Therefore

$$q^* = i \sum_3^4 d_k^* \begin{pmatrix} i\phi_k(\infty) \\ N_k^* \\ L_k^* \end{pmatrix}$$

In this notation

$$Q^*v^* = iq^*$$

$$\text{Let } r_k = \begin{pmatrix} i\phi_k(\infty) \\ N_k^* \\ L_k^* \end{pmatrix}$$

To determine the asymptotic properties of v^*, it is necessary first to determine the asymptotic properties of Q^*, r_3, and r_4, since we may write:

$$Q^*v^* = -d_3^*r_3 - d_4^*r_4$$

In Appendix I we obtain

$$L_3^* \to \frac{\pi}{8\alpha^3}\left(1 + \frac{7\Omega i}{8\alpha^2}\right)$$

$$L_4^* \to \frac{\pi}{16\alpha^3}\left(1 + \frac{\Omega i}{\alpha^2}\right)$$

From the results of Appendices II and III we obtain

$$N_3^* \to -ie^{i\Omega}K_1(i\Omega)$$

$$N_4^* \to ie^{i\Omega}\left(\frac{K_o(i\Omega)}{2} - \frac{i}{\Omega}K_1(i\Omega)\right)$$

In Appendix III we obtain

$$\phi_3(\infty) \to \frac{1}{\alpha}$$

$$\phi_4(\infty) \to -\frac{1}{2\alpha}$$

Furthermore

$$Q^* = \begin{pmatrix} i\phi_1(\infty) & i\phi_2(\infty) & i\phi_\alpha(\infty) \\ N_1^* & N_2^* & N_\alpha^* \\ L_1^* & L_2^* & L_\alpha^* \end{pmatrix}$$

Where

$$N_k^* = J_{k,c} + i(J_{k,s} - \mathcal{V}_k(\infty))$$

$$L_k^* = I_{k,c} + iI_{k,s}$$

N_α^* and L_α^* are similarly defined.

In Appendix I we obtain

$$L_1^* \to \frac{3\pi}{8\alpha^{5/2}}\left(1 + \frac{35}{24}\frac{\Omega}{\alpha^2}\, i\right)$$

$$L_2^* \to \frac{\pi}{8\alpha^{3/2}}\left(1 + \frac{5}{8}\frac{\Omega}{\alpha^2}\, i\right)$$

$$L_\alpha^* \to \frac{3\pi}{8\alpha^2}\left(1 + \frac{35}{8}\frac{\Omega}{\alpha^2}\, i\right)$$

From the results of Appendices II and III we obtain

$$N_1^* \to \frac{1}{2\alpha^{3/2}}\,\Omega e^{i\Omega}(K_o(i\Omega) + K_1(i\Omega))$$

$$N_2^* \to \frac{1}{2\alpha^{1/2}}\,\Omega e^{i\Omega}(K_o(i\Omega) + K_1(i\Omega))$$

$$N^* \to \frac{0}{\alpha}\,\Omega e^{i\Omega}(K_o(i\Omega) + K_1(i\Omega))$$

In Appendix III we obtain

$$\phi_1(\infty) \to \alpha^{-1/2}$$

$$\phi_2(\infty) \to -\alpha^{1/2}$$

$$\phi_\alpha(\infty) \to 1$$

Let

$$u_1 = \frac{d^*_{1,s} + id^*_{1,c}}{\alpha^{3/2}}$$

$$u_2 = \frac{d^*_{2,s} + id^*_{2,c}}{\alpha^{1/2}}$$

$$u_\alpha = \frac{\alpha_s + i\alpha_c}{\alpha}$$

$$\Gamma = \frac{K_1}{K_o + K_1} \quad \text{(Theodorsen's function)}$$

The three equations become

$$u_1 - u_2 + u_\alpha = 0$$

$$u_1 + u_2 = -(d_3^*(-\frac{2i}{\Omega}\Gamma) + d_4^*(\frac{i}{\Omega})(1-\Gamma(1+\frac{2i}{\Omega}))$$

$$= \frac{i}{\Omega}(2d_3^*\,\Gamma - d_4^*(1-\Gamma(1+2\,\frac{i}{\Omega})))$$

$$3u_1 + u_2 - 3u_\alpha = 0$$

Since d_3^*, d_4^* and Γ are independent of α and

$$\begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & -3 \end{vmatrix} = -8$$

u_1, u_2, u_α are all finite as $\alpha \to \infty$.

To obtain the pressure distribution on the hydrofoil, and also the lift and moment, it is sufficient to obtain $u_1 + u_2$ only since $\alpha u_\alpha \ \delta_\alpha \to 0$ as $\alpha \to 0$ while

$$\alpha^{3/2} u_1\delta_1 + \alpha^{1/2} u_2\delta_2 \to (u_1 + u_2)\sqrt{\frac{\ell-x}{\ell+x}} \text{ ; (see Appendix IV).}$$

Therefore the (normalized) net unsteady pressure δ on the hydrofoil (as $\alpha \to \infty$) is given by

$$\delta = (u_1 + u_2) \sqrt{\frac{\ell-x}{\ell+x}} - \frac{2\sqrt{\ell^2-x^2}}{\ell} d_3^* - \frac{x\sqrt{\ell^2-x^2}}{\ell^2} d_4^*$$

Similarly the (normalized) unsteady lift and moment are given by (see Appendix IV)

$$f^* = -\frac{1}{2\ell}\left(\sum_1^4 d_k^* R_k + \alpha^* R_\alpha\right)$$

$$\to \pi(u_1 + u_2 - d_3^*)$$

$$= \pi\left[\left(\frac{2i\Gamma}{\Omega} - 1\right)d_3^* - \left(\frac{i}{\Omega} + \left(\frac{2}{\Omega^2} - \frac{i}{\Omega}\right)\Gamma\right)d_4^*\right]$$

$$m^* = -\frac{1}{2\ell^2}\left(\sum_1^4 d_k^* S_k + \alpha^* S_\alpha\right)$$

$$\to \pi\left(\frac{u_1 + u_2}{2} - d_3^* - \frac{d_4^*}{8}\right)$$

$$= \pi\left[\left(\frac{i\Gamma}{\Omega} - 1\right)d_3^* - \left(\frac{1}{8} + \frac{i}{2\Omega} + \left(\frac{1}{\Omega^2} - \frac{i}{2\Omega}\right)\Gamma\right)d_4^*\right]$$

To obtain the limit of the fractional cavity length change we write from the first and third of the simultaneous equations,

$$u_\alpha = \frac{u_1 + u_2}{2}$$

Let $\omega^* = \omega_s^* + i\omega_c^*$ (see Section 6.)

$$\text{The } \omega^* = -\,2\alpha\omega(\alpha_s + i\alpha_c)$$

$$= -\,2\alpha^2\omega u_\alpha$$

$$= -\,\frac{2\alpha^2}{\alpha^2 + 1}\,u_\alpha$$

$$\rightarrow -\,2u_\alpha$$

Therefore as $\alpha \rightarrow \infty$

$$\omega^* \rightarrow -(u_1 + u_2)$$

$$= -\,\frac{2i\Gamma}{\Omega}\,d_3^* + \left(\frac{i}{\Omega} + \left(\frac{2}{\Omega^2} - \frac{i}{\Omega}\right)\Gamma\right)d_4^*$$

For the unit heave solution $d_3^* = -\,i\Omega^2$, $d_4^* = 0$

Therefore:

$$f^* = \pi(2\Omega\Gamma + i\Omega^2)$$

$$m^* = \pi(\Omega\Gamma + i\Omega^2)$$

$$\omega^* = -2\Omega\Gamma$$

or

$$f_s^* = \pi(2\Omega F)$$

$$f_c^* = \pi(2\Omega G + \Omega^2)$$

$$m_s^* = \pi\Omega F$$

$$m_c^* = \pi(\Omega G + \Omega^2)$$

$$\omega_s^* = -2\Omega F$$

$$\omega_c^* = -2\Omega G$$

where $\Gamma = F + iG$ (F,G real). This is Theodorsen's result.[2]
For the unit pitch solution, $d_3^* = -2\Omega$, $d_4^* = -i\Omega^2$.
Therefore

$$f^* = \pi(\Omega + \Omega\Gamma - 2i\Gamma)$$

$$m^* = \pi(\frac{3\Omega}{2} + \frac{i\Omega^2}{8} - i\Gamma + \frac{\Omega\Gamma}{2})$$

$$\omega^* = \Omega - \Omega\Gamma + 2i\Gamma$$

or

$$f_s^* = \pi(\Omega + \Omega F + 2G)$$

$$f_c^* = \pi(\Omega G - 2F)$$

$$m_s^* = \pi(\frac{3\Omega}{2} + G + \frac{\Omega F}{2})$$

$$m_c^* = \pi(\frac{\Omega^2}{8} - F + \frac{\Omega G}{2})$$

$$\omega_s^* = \Omega - \Omega F - 2G$$

$$\omega_c^* = -\Omega G + 2F$$

This is Theodorsen's result. [2]

f_s^*, f_c^*, m_s^*, m_c^*, ω_s^*, ω_c^* were calculated in both cases (unit heave and unit pitch) for $\Omega = 0$, .05, .1, .3 and from .5 to 4. in steps of .5 . The results are included in Table I (under $\omega_o = 0$).

Since to determine the limiting solution, the particular term $u_1 + u_2$ is sufficient, rather than the individual u_1, u_2, u_α, we see that only the second equation stated above is required. This equation is simply the statement that the vertical component of the velocity on the foil is determined by the foil motion. Since all physical models in this context include this condition and drop one of the remaining three (pressure at infinity, closure, continuity of the vertical component of velocity in the wake), we see that this limiting behaviour sheds no light on the question of the proper choice of model.

5. ZERO-FREQUENCY LIMIT

We now perform a quasi-steady analysis to determine the behaviour of the solution in the limit of low reduced frequency. This is done by approximating the exact solution at each time by the steady solution appropriate to the configuration at that time. The solution obtained by this approximation is called the quasi-steady solution.

For heave motion the quasi-steady solution is essentially independent of time so that the zero frequency limit is zero for all unsteady quantities.

For pitch motion, the angle of attack varies, and a non-trivial result is obtained. If the cavity length is less than 3/4 of the foil length, the magnitude of the unsteady force, moment, and fractional cavity change are all finite. The quasi-steady nature of the behavior implies that these quantities are all in phase (modulo π) with the motion, i.e. the sine component is zero. At 3/4 cavity length the magnitudes of the unsteady terms are infinite, so that a physically meaningful phase analysis can not be made.

The steady state expression for α is given implicitly by:

$$M = \frac{-\sigma}{U^2}\left(\frac{\alpha}{(\sqrt{\alpha^2+1} + \alpha)^2}\right)$$

Therefore

$$\frac{dM}{d\alpha} = \frac{-\sigma}{U^2}\left(\frac{\sqrt{\alpha^2+1} - 2\alpha}{\sqrt{\alpha^2+1}\,(\sqrt{\alpha^2+1} + \alpha)^2}\right)$$

and

$$\frac{d\alpha}{dM} = \frac{U^2}{\sigma}\,\frac{\sqrt{\alpha^2+1}\,(\sqrt{\alpha^2+1} + \alpha)^2}{2\alpha - \sqrt{\alpha^2+1}}$$

The quasi-steady change in the length of the cavity is gotten from

$$\frac{\Delta\omega}{\omega} = -\, 2\omega\alpha\Delta\alpha \; , \quad \text{where}$$

$$\Delta\alpha = \frac{d\alpha}{dM} M_1$$

$$= \frac{U^2}{\sigma} \; \frac{\sqrt{\alpha^2+1}\;(\sqrt{\alpha^2+1} + \alpha)^2}{2\alpha - \sqrt{\alpha^2+1}} M_1$$

$$= \frac{-\alpha\sqrt{\alpha^2+1}}{2\alpha - \sqrt{\alpha^2+1}} \; \frac{M_1}{M_o}$$

Therefore

$$\frac{\Delta\omega}{\omega} = \frac{2\alpha^2}{\sqrt{\alpha^2+1}\;(2\alpha - \sqrt{\alpha^2+1}} \; \frac{M_1}{M_o}$$

or

$$\omega_c^* = \frac{2\alpha^2}{\sqrt{\alpha^2+1}\;(2\alpha - \sqrt{\alpha^2+1}} \quad \text{in the quasi-steady limit.}$$

The quasi-steady force term is derived as follows:

$$\frac{f}{d\ell\sigma} = \pi(\sqrt{\alpha^2+1} - \alpha)$$

$$\frac{1}{d\ell\sigma}\,\frac{df}{d\alpha} = \pi\left(\frac{\alpha - \sqrt{\alpha^2+1}}{\sqrt{\alpha^2+1}}\right)$$

$$\frac{1}{d\ell U^2} \; \frac{df}{dM} = -\,\pi \, \frac{(\sqrt{\alpha^2+1} + \alpha)}{2\alpha - \sqrt{\alpha^2+1}}$$

$$\Delta f = \frac{df}{dM} M_1$$

$$= -(d\ell U^2 M_o) \frac{\pi(\sqrt{\alpha^2+1} + \alpha)}{2\alpha - \sqrt{\alpha^2+1}} \frac{M_1}{M_o}$$

or

$$f_c^* = - \frac{\pi(\sqrt{\alpha^2+1} + \alpha)}{2\alpha - \sqrt{\alpha^2+1}} \qquad \text{in the quasi-steady limit.}$$

The quasi-steady moment term follows from

$$\frac{m}{\sigma d\ell^2} = \frac{\pi}{4(\alpha^2 + 1)^{3/2}} (\alpha\sqrt{\alpha^2+1} + \frac{4}{(\sqrt{\alpha^2+1} + \alpha)^2})$$

$$\frac{1}{\sigma d\ell^2} \frac{dm}{d\alpha} = \pi(\frac{1 - \alpha^2}{4(\alpha^2+1)^2} - \frac{3\alpha + 2\sqrt{\alpha^2+1}}{(\sqrt{\alpha^2+1} + \alpha)^2(\alpha^2+1)^{5/2}})$$

$$\frac{dm}{dM} = (U^2 d\ell^2)\pi(\frac{3\alpha + 2\sqrt{\alpha^2+1}}{(\sqrt{\alpha^2+1} - 2\alpha)(\alpha^2+1)^2} + \frac{(\alpha^2-1)(\sqrt{\alpha^2+1} + \alpha)^2}{4(\sqrt{\alpha^2+1} - 2\alpha)(\alpha^2+1)^{3/2}})$$

$$\Delta m = \frac{dm}{dM} M_1$$

Therefore, in the quasi-steady limit

$$m_c^* = \frac{\pi}{(\sqrt{\alpha^2+1} - 2\alpha)(\alpha^2+1)^2}[\alpha+2(\sqrt{\alpha^2+1}+\alpha) + \frac{(\alpha^2-1)\sqrt{\alpha^2+1}(\sqrt{\alpha^2+1} + \alpha)^2}{4}]$$

The results associated with $\Omega = 0$ in Table I are based on these formulas.

6. DISCUSSION

In Table I we have tabulated the cosine and sine components of the normalized unsteady force, moment and cavity length, f_c^*, f_s^*, m_c^*, m_s^*, ω_c^*, ω_s^*, due to unit heave and due to unit pitch. We may obtain the total force f, moment m, and cavity length ω from the following:

$$\text{Force} \quad f = f_o + C_f(f_c^* \cos\nu t + f_s^* \sin\nu t)$$

$$\text{Moment} \quad m = m_o + C_m(m_c^* \cos\nu t + m_s^* \sin\nu t)$$

$$\text{Cavity Length} \quad \omega = \omega_o + C_\omega(\omega_c^* \cos\nu t + \omega_s^* \sin\nu t)$$

For heaving motion of amplitude B_1:

$$C_f = dU^2B_1$$

$$C_m = d\ell U^2B_1$$

$$C_\omega = \omega_oB_1/M_o\ell$$

For pitching motion of amplitude M_1:

$$C_f = d\ell U^2M_1$$

$$C_m = d\ell^2U^2M_1$$

$$C_\omega = \omega_oM_1/M_o$$

The steady state terms are given by

$$f_o = -d\ell U^2M_o\pi\left(1 + \frac{\sqrt{\alpha^2+1}}{\alpha}\right)$$

$$m_o = -d\ell^2U^2M_o\pi\left(\frac{1}{\alpha(\alpha^2+1)^{3/2}} + \frac{(\sqrt{\alpha^2+1} + \alpha)^2}{4(\alpha^2+1)}\right)$$

$$\omega_o = 1/(\alpha^2+1)$$

UNSTEADY LIFT FORCE PERPENDICULAR TO FOIL DUE TO UNIT HEAVE

σ_o (Cavity Length) / Ω (Reduced Frequency)	0	.125	.25	.375	.5	.625	.75	
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	
.05	-.0332	-.0394	-.0519	-.0769	-.135	-.324	-1.52	
.1	-.0768	-.0889	-.116	-.171	-.288	-.599	-1.74	
.3	-.0553	-.0465	-.0876	-.199	-.429	-.901	-2.00	
.5	.312	.394	.382	.251	-.0470	-.628	-1.83	
1.0	2.51	2.95	3.11	2.97	2.48	1.55	-.0890	
1.5	6.38	7.41	7.85	7.70	6.98	5.66	3.51	f_c^*
2.0	11.8	13.7	14.5	14.4	13.4	11.7	9.05	
2.5	18.9	21.8	23.0	22.9	21.7	19.6	16.3	
3.0	27.5	31.7	33.4	33.3	31.8	29.2	25.0	
3.5	37.7	43.3	45.6	45.5	43.8	40.5	34.7	
4.0	49.5	56.8	59.6	59.5	57.4	53.2	45.2	
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	
.05	.286	.314	.353	.414	.517	.727	.706	
.1	.523	.570	.633	.722	.859	+1.08	1.01	
.3	1.25	1.35	1.47	1.61	1.79	1.99	2.01	
.5	1.88	2.03	2.20	2.39	2.59	2.81	2.90	
1.0	3.39	3.70	4.03	4.34	4.61	4.81	4.87	
1.5	4.91	5.41	5.94	6.39	6.67	6.76	6.64	f_s^*
2.0	6.45	7.17	7.96	8.55	8.81	8.75	8.53	
2.5	7.99	9.00	10.1	10.8	11.0	10.9	10.9	
3.0	9.54	10.9	12.4	13.2	13.4	13.4	13.8	
3.5	11.1	12.9	14.8	15.8	16.0	16.2	17.2	
4.0	12.7	14.9	17.3	18.4	18.8	19.4	20.5	

TABLE I

UNSTEADY LIFT FORCE PERPENDICULAR TO FOIL DUE TO UNIT PITCH

Ω (Reduced Frequency) \ ω_o (Cavity Length)	0	.125	.25	.375	.5	.625	.75	
0.0	-6.28	-6.98	-8.01	-9.68	-12.9	-22.5	-	
.05	-5.73	-6.30	-7.10	-8.30	-10.4	-14.6	-14.3	
.1	-5.28	-5.76	-6.39	-7.30	-8.63	-10.9	-10.4	
.3	-4.35	-4.70	-5.10	-5.59	-6.18	-6.37	-6.95	
.5	-3.99	-4.35	-4.72	-5.10	-5.51	-5.91	-6.03	
1.0	-3.70	-4.23	-4.66	-4.98	-5.17	-5.22	-5.06	f_c^*
1.5	-3.62	-4.45	-5.04	-5.34	-5.36	-5.14	-4.73	
2.0	-3.59	-4.84	-5.64	-5.93	-5.30	-5.40	-4.94	
2.5	-3.57	-5.37	-6.41	-6.70	-6.45	-5.99	-5.64	
3.0	-3.56	-6.02	-7.33	-7.61	-7.29	-6.82	-6.58	
3.5	-3.55	-6.77	-8.38	-8.65	-8.28	-7.81	-7.45	
4.0	-3.55	-7.63	-9.52	-9.79	-9.39	-8.61	-8.03	
0.0	0.0	0.0	0.0	0.0	0.0	0.0	-	
.05	-.521	-.647	-.397	-1.40	-2.55	-6.32	-30.4	
.1	-.507	-.632	-.909	-1.46	-2.64	-5.76	-17.2	
.3	.443	.455	.295	-.110	-.923	-2.57	-6.34	
.5	1.56	1.70	1.64	1.32	.650	-.617	-3.16	
1.0	4.21	4.61	4.63	4.43	3.82	2.76	.971	f_s^*
1.5	6.71	7.32	7.48	7.22	6.59	5.61	4.14	
2.0	9.14	9.25	10.1	9.84	9.23	8.34	7.07	
2.5	11.6	12.5	12.7	12.3	11.8	10.9	9.66	
3.0	13.9	15.0	15.1	14.8	14.2	13.3	11.8	
3.5	16.3	17.5	17.5	17.1	16.5	15.5	13.4	
4.0	18.7	19.9	19.7	19.3	18.7	17.4	14.8	

TABLE I (Continued)

UNSTEADY MOMENT DUE TO UNIT HEAVE

σ_o (Cavity Length) → / Ω (Reduced Frequency) ↓	0	.125	.25	.375	.5	.625	.75	
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	
.05	-.0127	-.0101	-.0123	-.0257	-.0719	-.261	-1.71	
.1	-.0227	-.0134	-.0149	-.0431	-.141	-.473	-1.95	
.3	.114	.175	.212	.177	-.0197	-.592	-2.22	
.5	.549	.689	.806	.815	.587	-.126	-1.97	
1.0	2.83	3.28	3.75	4.00	3.77	2.74	.231	m_c^*
1.5	6.72	7.67	8.68	9.30	9.14	7.82	4.70	
2.0	12.2	13.8	15.5	16.6	16.6	15.0	11.5	
2.5	19.3	21.7	24.3	25.9	26.0	24.4	20.5	
3.0	27.9	31.4	34.9	37.0	37.4	35.7	31.1	
3.5	38.1	42.8	47.3	50.0	50.7	49.0	43.1	
4.0	49.9	55.9	61.5	64.8	65.9	64.1	56.2	
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	
.05	.143	.126	.137	.189	.314	.605	.795	
.1	.261	.230	.248	.334	.526	.901	1.14	
.3	.627	.551	.596	.781	1.14	1.70	2.26	
.5	.939	.835	.915	1.20	1.70	2.44	3.26	
1.0	1.69	1.54	1.76	2.33	3.20	4.30	5.48	m_s^*
1.5	2.46	2.28	2.71	3.64	4.88	6.22	7.48	
2.0	3.22	3.06	3.80	5.17	6.75	8.24	9.60	
2.5	4.00	3.89	5.04	6.89	8.79	10.4	12.2	
3.0	4.77	4.78	6.45	8.80	11.0	12.9	15.5	
3.5	5.55	5.73	8.01	10.8	13.2	15.6	19.3	
4.0	6.33	6.76	9.72	13.0	15.6	18.5	23.1	

TABLE I (Continued)

UNSTEADY MOMENT DUE TO UNIT PITCH

Ω (Reduced Frequency) \ σ_c (Cavity Length)	0	.125	.25	.375	.5	.625	.75	
0.0	-3.14	-2.80	-3.10	-4.41	-7.81	-18.7	-	m_c^*
.05	-2.87	-2.53	-2.76	-3.80	-6.30	-12.2	-16.1	
.1	-2.64	-2.32	-2.50	-3.37	-5.32	-9.12	-11.6	
.3	-2.14	-1.88	-2.04	-2.69	-3.92	-5.85	-7.80	
.5	-1.90	-1.71	-1.91	-2.53	-3.58	-5.09	-6.72	
1.0	-1.46	-1.45	-1.85	-2.58	-3.51	-4.54	-5.46	
1.5	-.927	-1.18	-1.89	-2.82	-3.72	-4.41	-4.80	
2.0	-.222	-.830	-1.94	-3.11	-3.98	-4.44	-4.65	
2.5	.670	-.386	-1.97	-3.38	-4.24	-4.59	-4.92	
3.0	1.76	.163	-1.94	-3.57	-4.43	-4.81	-5.35	
3.5	3.03	.822	-1.82	-3.66	-4.55	-5.01	-5.59	
4.0	4.51	1.60	-1.59	-3.61	-4.59	-5.12	-5.38	
0.0	0.0	0.0	0.0	0.0	0.0	0.0	-	m_s^*
.05	-.103	-.0526	-.0946	-.362	-1.28	-5.05	-34.1	
.1	.0606	.157	.141	-.143	-1.12	-4.45	-19.3	
.3	1.16	1.39	1.52	1.38	.669	-1.33	-6.94	
.5	2.35	2.68	2.93	2.90	2.34	.738	-3.23	
1.0	5.24	5.83	6.30	6.43	5.98	4.63	1.76	
1.5	8.07	8.90	9.54	9.72	9.27	8.01	5.65	
2.0	10.9	11.9	12.7	12.8	12.3	11.2	9.28	
2.5	13.6	14.9	15.7	15.7	15.3	14.3	12.5	
3.0	16.4	17.8	18.5	18.5	18.0	17.1	15.2	
3.5	19.2	20.7	21.3	21.2	20.7	19.8	17.4	
4.0	21.9	23.6	24.0	23.8	23.4	22.3	19.3	

TABLE I (Continued)

UNSTEADY FRACTIONAL CAVITY CHANGE DUE TO UNIT HEAVE

σ_o (Cavity Length)	0	.125	.25	.375	.5	.625	.75	
Ω (Reduced Frequency)								
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	
.05	.0131	.0147	.0171	.0210	.0289	.0514	.164	
.1	.0345	.0384	.0440	.0523	.0669	.0997	.190	
.3	.108	.120	.135	.151	.173	.204	.252	
.5	.151	.174	.197	.220	.242	.267	.297	
1.0	.201	.269	.324	.363	.386	.393	.387	σ_c^*
1.5	.221	.359	.458	.509	.514	.484	.429	
2.0	.231	.462	.607	.648	.606	.512	.404	
2.5	.236	.584	.763	.761	.637	.472	.329	
3.0	.240	.722	.914	.829	.605	.384	.253	
3.5	.242	.874	1.05	.843	.522	.292	.233	
4.0	.244	1.04	1.15	.805	.418	.243	.294	
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	
.05	-.0909	-.0902	-.0901	-.0914	-.0956	-.106	-.0719	
.1	-.166	-.163	-.160	-.158	-.156	-.155	-.101	
.3	-.399	-.380	-.359	-.334	-.305	-.265	-.186	
.5	-.598	-.562	-.521	-.474	-.420	-.353	-.254	
1.0	-1.08	-.996	-.897	-.784	-.661	-.528	-.379	
1.5	-1.56	-1.43	-1.25	-1.05	-.841	-.644	-.455	σ_s^*
2.0	-2.05	-1.84	-1.56	-1.25	-.967	-.726	-.526	
2.5	-2.54	-2.25	-1.83	-1.41	-1.06	-.818	-.644	
3.0	-3.04	-2.64	-2.06	-1.53	-1.17	-.961	-.834	
3.5	-3.53	-3.01	-2.25	-1.64	-1.31	-1.18	-1.08	
4.0	-4.03	-3.35	-2.40	-1.76	-1.51	-1.45	-1.32	

TABLE I (Continued)

UNSTEADY FRACTIONAL CAVITY CHANGE DUE TO UNIT PITCH

σ_e (Cavity Length) / Ω (Reduced Frequency)	0	.125	.25	.375	.5	.625	.75	
0.0	2.00	2.01	2.05	2.15	2.41	3.34	-	
.05	1.82	1.81	1.81	1.83	1.92	2.14	1.46	
.1	1.68	1.65	1.63	1.60	1.58	1.56	1.03	
.3	1.38	1.32	1.24	1.16	1.05	.914	.642	
.5	1.27	1.19	1.10	.997	.876	.731	.523	
1.0	1.18	1.06	.936	.800	.659	.514	.358	
1.5	1.15	.992	.819	.652	.300	.369	.254	σ_c^*
2.0	1.14	.921	.699	.511	.371	.278	.216	
2.5	1.14	.843	.573	.385	.286	.251	.242	
3.0	1.13	.755	.448	.289	.255	.281	.302	
3.5	1.13	.658	.332	.230	.272	.340	.351	
4.0	1.13	.555	.231	.211	.323	.396	.355	
0.0	0.0	0.0	0.0	0.0	0.0	0.0	-	
.05	.266	.295	.341	.417	.573	1.02	3.27	
.1	.361	.396	.446	.525	.668	.994	1.91	
.3	.459	.484	.515	.556	.615	.710	.867	
.5	.502	.518	.534	.550	.570	.599	.640	
1.0	.661	.659	.644	.614	.571	.518	.457	
1.5	.866	.846	.790	.701	.589	.468	.350	σ_s^*
2.0	1.09	1.05	.930	.762	.574	.394	.245	
2.5	1.32	1.24	1.05	.788	.528	.314	.169	
3.0	1.56	1.44	1.14	.783	.473	.260	.150	
3.5	1.80	1.62	1.20	.760	.433	.255	.190	
4.0	2.05	1.79	1.24	.735	.427	.302	.261	

TABLE I (Continued)

In these formulas, f is positive upwards perpendicular to the foil, m is counter-clockwise around the leading edge (with the flow going from left to right), and ω is cavity length/chord.

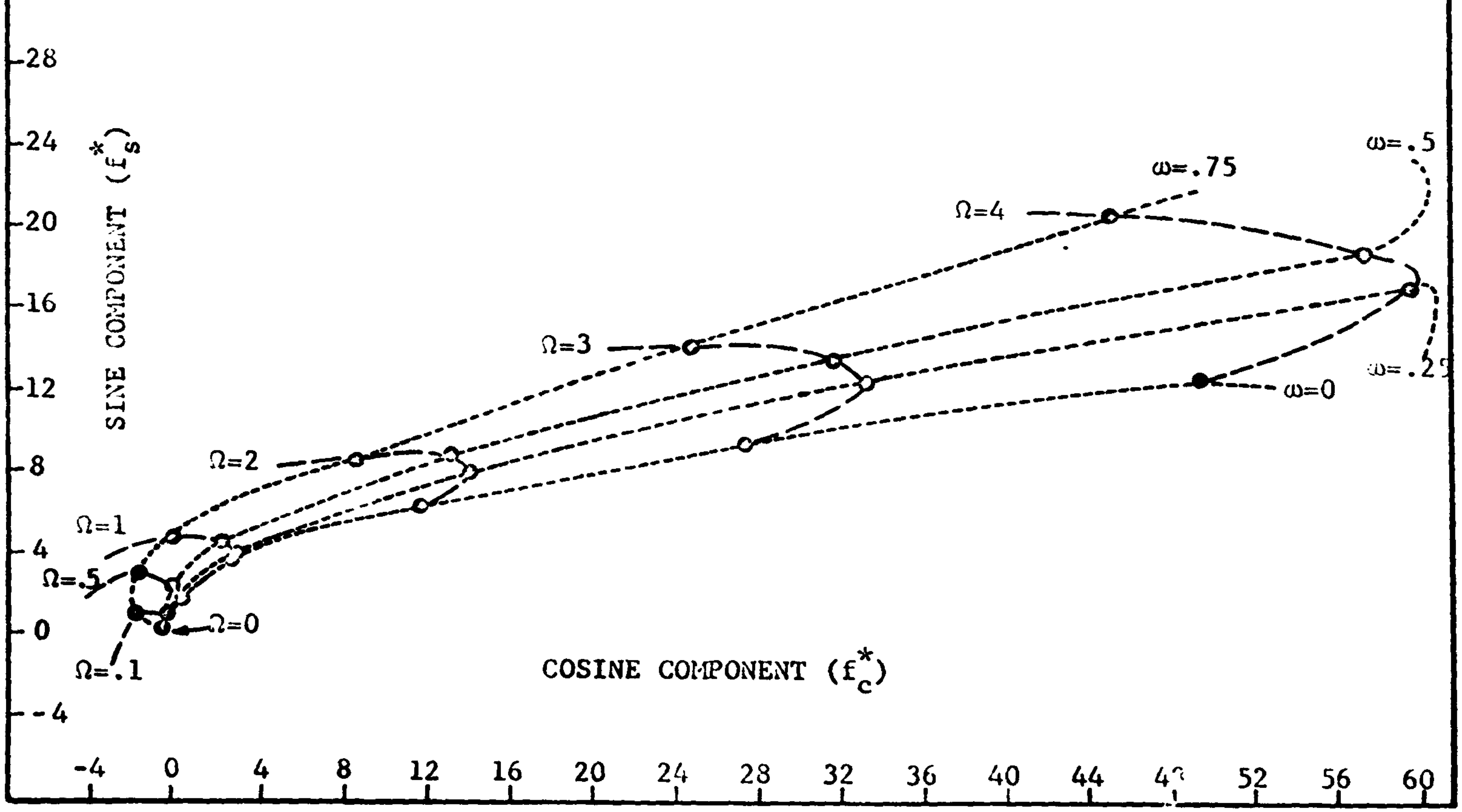
We now call attention to some overall features of the data. In figures 1, 2, and 3 are examples of the type of plot which indicates the general behaviour of each of the unsteady terms, constructed by plotting the sine coefficient ("s" subscript) against the cosine coefficient ("c" subscript) and indicating contours of constant frequency and constant cavity length.

All heave functions have the following low frequency behaviour: amplitude goes to zero, while the phase tends to $\pm\ \pi/2$, except for the critical 3/4 cavity length. For this case the phase cannot be precisely evaluated, but it appears to be 0 (modulo π). For high frequencies, the force and moment increase in amplitude and tend to become fairly constant in phase, the value depending on the cavity length. The unsteady cavity length term increases in magnitude at a roughly constant phase up to some critical frequency (dependent on the cavity length) and then more slowly increases in amplitude and undergoes a rapid phase change.

The unsteady pitch terms have significantly different behaviours at low and high frequency (Ω = .5 appears to be the transition). At low frequencies except at the critical 3/4 cavity length, the amplitude tends to the value predicted from

FORCE DUE TO UNIT HEAVE

FIGURE 1.



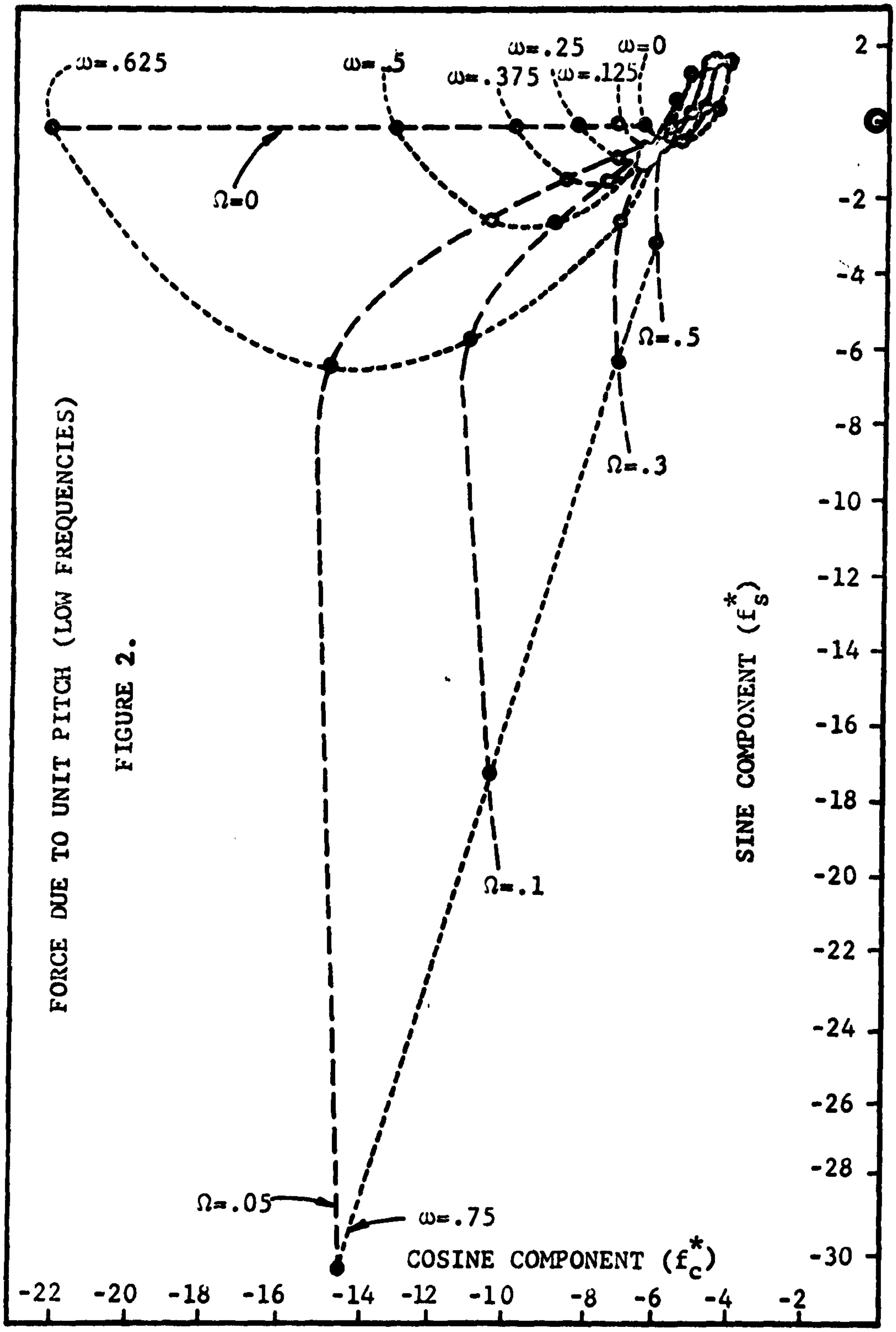
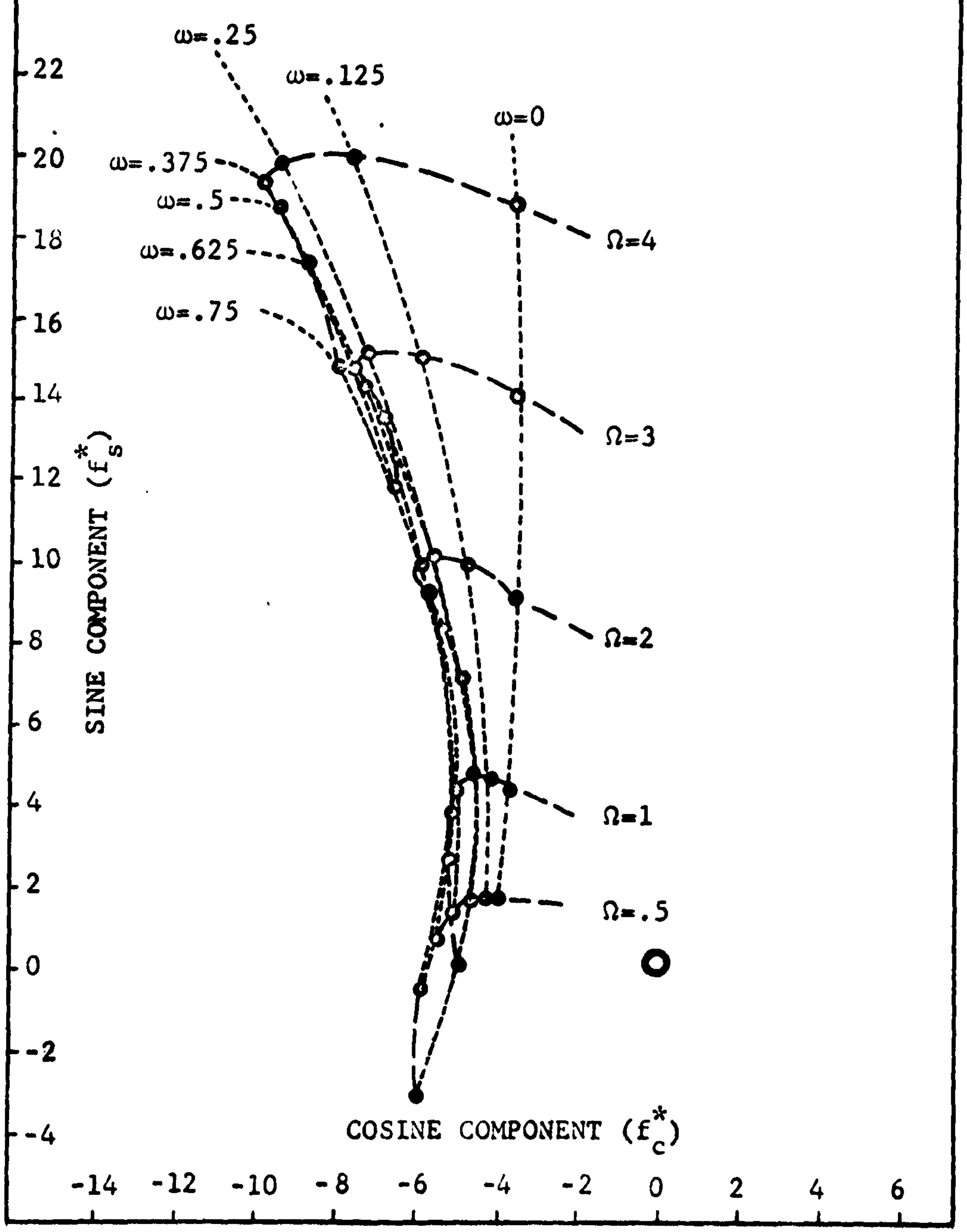


FIGURE 2.

FORCE DUE TO UNIT PITCH (LOW FREQUENCIES)

FORCE DUE TO UNIT PITCH (HIGH FREQUENCIES)

FIGURE 3.



the steady state theory, and the phase tends to 0 (modulo π). For the 3/4 cavity case, the amplitude becomes ∞, as expected from steady theory while the phase is again hard to predict, but seems to approach $\pm\pi/2$. At high frequencies, the behaviour of the force and moment in general resembles that of high frequency heave motion, i.e. increasing amplitude, with no significant phase effect.

The cavity length term exhibits an unusual behaviour. As we go from zero to high frequency, for each steady cavity length, the amplitude first decreases at roughly constant phase, then begins to change phase while the amplitude reaches a minimum and begins to increase. This strange behaviour may be worth further study.

7. CONCLUSION

Numerical calculations have been made to obtain the quantities which describe the unsteady behaviour of a two-dimensional flat-plate hydrofoil in an infinite fluid without gravity subject to oscillatory heaving or pitching, based on a particular physical model. The limiting behaviour for zero cavity length was obtained by an asymptotic development, while a quasi-steady analysis was carried out to obtain the zero-frequency behaviour. For zero cavity length we found that the limiting behaviour does not depend on the particular physical model, i.e. dropping the closure condition, that was selected. The consistency of the quasi-steady results (for the zero-frequency limit) with the numerical results tends to support our choice of model.

It would be of interest to carry out an asymptotic expansion, instead of a quasi-steady analysis, in the zero-frequency limit to determine if all models are consistent in this case also. Furthermore since experimental evidence indicates that the low frequency (reduced frequency less than .5) behaviour is important, further study of the limiting behaviour would have independent interest.

Since it has by no means been conclusively demonstrated that this particular physical model (omitting the cavity closure condition) is correct, although it seems the most reasonable, it would be of interest to work out all other models

which are possible in the same context, i.e. as solutions to Laplace's equation in linearized theory. Specifically the unsteady closure condition would be set up as a fourth condition and solutions would be obtained for the models obtained by dropping the continuity of the vertical component of velocity and dropping the condition of having the acceleration potential go to zero at infinity. In addition a third case may be constructed by introducing an unsteady singularity at the trailing edge of the foil (i.e. dropping the unsteady Kutta condition) and then using all four previously mentioned boundary conditions.

It would also be desirable to extend the analysis to include

1) non-zero thickness and camber
2) finite span
3) curved, surface-piercing foils
4) gravity and free surface effects,

and to perform a hydroelastic analysis in each of these cases.

APPENDIX I - Asymptotic Values of the I Integrals (see p.22 and Appendix IV of [1])

The $I_{k,c}$ and $I^*_{k,c}$ are of the form

$$\int_{-\infty}^{\infty} \tilde{\psi}_k \frac{\xi(\xi^2+\alpha)}{P^2} \cos\left(\frac{2\Omega}{P}\right) d\xi$$

where $P = (\xi^2 + \alpha)^2 + 1$ (and where $\frac{\partial\tilde{\psi}_k}{\partial\alpha}$ is used instead of $\tilde{\psi}_k$ for $I^*_{k,c}$.)

For all values of ξ and α, $P \geq \alpha^2 + 1$.

Therefore $\frac{2\Omega}{P} \leq \frac{2\Omega}{\alpha^2+1}$, so that $\cos(\frac{2\Omega}{P}) \sim 1$, for large α and fixed Ω, independent of ξ .

Consequently the asymptotic value of the integral may be given by the asymptotic value of

$$\int_{-\infty}^{\infty} \tilde{\psi}_k \frac{\xi(\xi^2 + \alpha)}{P^2} d\xi$$

Similarly $\sin\left(\frac{2\Omega}{P}\right) \sim \frac{2\Omega}{P}$, so that

$$I_{k,s} \sim 2\Omega \int_{-\infty}^{\infty} \tilde{\psi}_k \frac{\xi(\xi^2 + \alpha)}{P^3} d\xi$$

Since all the $I_{k,c}$, $I_{k,s}$ etc. are expressible in terms of I_1, its asymptotic value will first be given

$$I_1 = \frac{\pi B}{\sqrt{\alpha^2+1}}$$

$$= \frac{\pi}{\sqrt{\alpha^2+1}} \sqrt{\frac{1}{2(\sqrt{\alpha^2+1} + \alpha)}}$$

$$\to \frac{\pi}{2\alpha^{3/2}}$$

$$I_{1,c} \sim \int_{-\infty}^{\infty} \frac{\xi^2 + \alpha}{P^2} d\xi$$

$$= J_2 + \alpha I_2$$

$$= J_2 + \frac{\alpha}{1 + \alpha^2} (\frac{3}{4} I_1 - \alpha J_2)$$

$$= \frac{J_2}{1 + \alpha^2} + \frac{3}{4} \frac{\alpha}{1 + \alpha^2} I_1$$

$$= \frac{J_1}{4(1 + \alpha^2)} + \frac{\alpha I_1}{2(1 + \alpha^2)}$$

$$= \frac{I_1}{4(1 + \alpha^2)} (\sqrt{\alpha^2 + 1} + 2\alpha)$$

$$\to \frac{3}{4\alpha} I_1$$

$$= \frac{3\pi}{8\alpha^{5/2}}$$

$$I_{1,s} \sim 2\Omega \int_{-\infty}^{\infty} \frac{\xi^2 + \alpha}{P^3} d\xi$$

$$= 2\Omega(J_3 + \alpha I_3)$$

$$= 2\Omega(J_3 + \frac{\alpha}{1 + \alpha^2} (\frac{7}{8} I_2 - \alpha J_3))$$

$$= 2\Omega(\frac{7}{8} \frac{\alpha I_2}{1 + \alpha^2} + \frac{1}{1 + \alpha^2} (- \frac{\alpha I_2}{8} + \frac{5}{8} J_2))$$

$$= \frac{2\Omega}{1 + \alpha^2} \left[\frac{3}{4} \alpha I_2 + \frac{5}{8} J_2\right]$$

$$= \frac{2\Omega}{1 + \alpha^2} (\frac{3}{4} \left[\frac{I_1}{4(1 + \alpha^2)} (\sqrt{\alpha^2+1} + 2\alpha)\right] - \frac{J_2}{8})$$

$$I_{1,s} \sim \frac{2\Omega}{1+\alpha^2}\left(\frac{3}{4}\left(\frac{\sqrt{\alpha^2+1}+2\alpha}{4(1+\alpha^2)}+\frac{1}{8}\left(\frac{\alpha}{4}-\frac{\sqrt{1+\alpha^2}}{4}\right)\right)\right) I_1$$

$$= \frac{2\Omega}{1+\alpha^2}\left(\frac{3}{16}\left(\frac{\sqrt{\alpha^2+1}+2\alpha}{1+\alpha^2}\right)-\frac{1}{32(\sqrt{\alpha^2+1}+\alpha)}\right) I_1$$

$$\rightarrow \frac{2\Omega}{\alpha^2}\left(\frac{9}{16\alpha}-\frac{1}{64\alpha}\right)\frac{\pi}{2\alpha^{3/2}}$$

$$= \frac{35\pi\Omega}{64\alpha^{9/2}}$$

$$I_{2,c} \sim \int_{-\infty}^{\infty} \frac{\xi^2(\xi^2+\alpha)}{P^3}\, d\xi$$

$$= \frac{I_1}{4}$$

$$\rightarrow \frac{\pi}{8\alpha^{3/2}}$$

$$I_{2,s} \sim 2\Omega \int_{-\infty}^{\infty} \frac{\xi^2(\xi^2+\alpha)}{P^3}\, d\xi$$

$$= \frac{2\Omega I_2}{8}$$

$$= \frac{\Omega}{4(1+\alpha^2)}\left(\frac{3}{4} I_1 - \alpha J_2\right)$$

$$= \frac{\Omega}{4(1+\alpha^2)}\left(\frac{3}{4}+\alpha\left(\frac{\alpha-\sqrt{\alpha^2+1}}{4}\right)\right) I_1$$

$$= \frac{\Omega}{4(1+\alpha^2)}\left(\frac{3}{4}-\frac{\alpha}{4(\alpha+\sqrt{\alpha^2+1})}\right) I_1$$

$$\rightarrow \frac{5\Omega}{32\alpha^2}\cdot\frac{\pi}{2\alpha^{3/2}}$$

$$= \frac{5\pi\Omega}{64\alpha^{7/2}}$$

$I^*_{k,c}$ and $I^*_{k,s}$ may be obtained from $I_{k,c}$ and $I_{k,s}$ by differentiating with respect to α, noting L'Hopital's rule.

$$I^*_{1,c} \to -\frac{15}{16}\frac{\pi}{\alpha^{7/2}}$$

$$I^*_{1,s} \to -\frac{315}{128}\frac{\pi\Omega}{\alpha^{11/2}}$$

$$I^*_{2,c} \to -\frac{3}{16}\frac{\pi}{\alpha^{5/2}}$$

$$I^*_{2,s} \to -\frac{35}{128}\frac{\pi\Omega}{\alpha^{9/2}}$$

Therefore

$$I_{\alpha,c} \to -\frac{3}{8}\frac{\pi}{\alpha^2}$$

$$I_{\alpha,s} \to -\frac{105}{64}\frac{\pi\Omega}{\alpha^4}$$

$$I_{3,c} \sim \frac{2A}{\sqrt{\alpha^2+1}} \int \frac{\xi^2(\xi^2+\alpha)(\xi^2+E)}{P^3}\, d\xi$$

$$= \frac{2A}{\sqrt{\alpha^2+1}} \left(E\,\frac{I_2}{8} + \frac{3J_2}{8}\right)$$

$$= \frac{A}{4\alpha\sqrt{\alpha^2+1}} \left((3\alpha - E)\, J_2 + E\frac{(\sqrt{\alpha^2+1} + 2\alpha)}{4(\alpha^2+1)}\right) I_1$$

$$= \frac{A}{4\alpha\sqrt{\alpha^2+1}} \left((\alpha + \sqrt{\alpha^2+1})\left(\frac{\sqrt{\alpha^2+1} - \alpha}{4}\right) + \frac{(2\alpha-\sqrt{\alpha^2+1})(2\alpha+\sqrt{\alpha^2+1})}{4(\alpha^2+1)}\right) I_1$$

$$= \frac{A}{16\alpha\sqrt{\alpha^2+1}} \left(1 + \frac{3\alpha^2 - 1}{\alpha^2+1}\right) I_1$$

$$= \frac{\pi}{32\alpha(\alpha^2+1)} \left(\frac{4\alpha^2}{\alpha^2+1}\right)$$

$$\to \frac{\pi}{8\alpha^3}$$

$$I_{3,s} \sim \frac{4A\Omega}{\sqrt{\alpha^2+1}} \int \frac{\xi^2(\xi^2+\alpha)(\xi^2+E)}{P^4}\, d\xi$$

$$= \frac{4A\Omega}{12\sqrt{\alpha^2+1}} (EI_3 + 3J_3)$$

$$= \frac{A\Omega}{3\alpha\sqrt{\alpha^2+1}} \left(\frac{E}{1+\alpha^2}\left(\frac{3}{16}\left(\frac{\sqrt{\alpha^2+1}+2\alpha}{1+\alpha^2}\right) - \frac{1}{32(\sqrt{\alpha^2+1}+\alpha)}\right) I_1 + (\alpha+\sqrt{\alpha^2+1}) J_3\right)$$

$$J_3 = -\frac{\alpha I_2}{8} + \frac{5}{8} J_2$$

$$= -\frac{1}{8}\left(\frac{\sqrt{\alpha^2+1}+2\alpha}{4(1+\alpha^2)}\right) I_1 + \frac{3}{4}\left(\frac{\sqrt{\alpha^2+1}-\alpha}{4}\right) I_1$$

$$I_{3,s} \sim \frac{A\Omega I_1}{3\alpha\sqrt{\alpha^2+1}} \left(\frac{2\alpha-\sqrt{\alpha^2+1}}{1+\alpha^2}\left(\frac{3}{16}\left(\frac{\sqrt{\alpha^2+1}+2\alpha}{1+\alpha^2}\right) - \frac{1}{32(\sqrt{\alpha^2+1}+\alpha)}\right)\right.$$

$$\left. + (\alpha+\sqrt{\alpha^2+1})\left(\frac{3(\sqrt{\alpha^2+1}-\alpha)}{16} - \frac{\sqrt{\alpha^2+1}+2\alpha}{32(1+\alpha^2)}\right)\right)$$

$$= \frac{A\Omega I_1}{96\alpha\sqrt{\alpha^2+1}} \left(6\left(\frac{3\alpha^2-1}{(\alpha^2+1)^2}+1\right)\right.$$

$$\left. - \frac{(\sqrt{\alpha^2+1}-\alpha)(2\alpha-\sqrt{\alpha^2+1})+(\sqrt{\alpha^2+1}+\alpha)(2\alpha+\sqrt{\alpha^2+1})}{1+\alpha^2}\right)$$

$$= \frac{A\Omega I_1}{96\alpha\sqrt{\alpha^2+1}} \left(6\left(\frac{3\alpha^2-1}{(\alpha^2+1)^2} + 1\right) - \frac{6\alpha\sqrt{\alpha^2+1}}{\alpha^2+1}\right)$$

$$= \frac{\Omega\pi}{32\alpha(\alpha^2+1)} \left(\frac{3\alpha^2-1}{(\alpha^2+1)^2} + \frac{1}{\sqrt{\alpha^2+1}\,(\sqrt{\alpha^2+1}+\alpha)}\right)$$

$$= \frac{7\Omega\pi}{64\alpha^5}$$

$$I_{4,c} \sim \frac{2A}{\sqrt{\alpha^2+1}} \int \frac{\xi^2(\xi^2+\alpha)(\xi^2+E)}{P^4}\, d\xi$$

$$= \frac{A}{2(\alpha^2+1)} \int_{-\infty}^{\infty} \frac{\xi^2(\xi^2+\alpha)(\alpha\xi^2-F)}{P^3}\, d\xi - \frac{3\alpha^2+1}{4(\alpha^2+1)} (I_{3,c}) \text{ (See Footnote)}$$

$$= M_1 + M_2 + M_3$$

$$M_1 = \frac{2A}{12\sqrt{\alpha^2+1}} (EI_3 + 3J_3)$$

$$= \frac{\pi}{64\alpha(\alpha^2+1)} \left(\frac{3\alpha^2-1}{(\alpha^2+1)^2} + \frac{1}{\sqrt{\alpha^2+1}\,(\sqrt{\alpha^2+1}+\alpha)}\right)$$

$$M_2 = \frac{A}{2(\alpha^2+1)} \left(\frac{3\alpha J_2}{8} - \frac{FI_2}{8}\right)$$

$$= \frac{A}{16\alpha(\alpha^2+1)} \left((3\alpha^2+F)J_2 - F\left(\frac{\sqrt{\alpha^2+1}+2\alpha}{4(1+\alpha^2)}\right) I_1\right)$$

$$= \frac{AI_1}{16\alpha(\alpha^2+1)} \left((3\alpha^2+F)\left(\frac{\sqrt{\alpha^2+1}-\alpha}{4}\right) - \frac{F}{4}\left(\frac{\sqrt{\alpha^2+1}+2\alpha}{4(\alpha^2+1)}\right)\right)$$

$$= \frac{\pi}{128\alpha(\alpha^2+1)^{3/2}} \left(\frac{3\alpha^2}{\sqrt{\alpha^2+1}+\alpha}\right.$$

$$\left. + (\alpha\sqrt{\alpha^2+1} - 2(\alpha^2+1))\left(\sqrt{\alpha^2+1} - \alpha - \frac{(\sqrt{\alpha^2+1}+2\alpha)}{\alpha^2+1}\right)\right)$$

$$= \frac{\pi}{128\alpha(\alpha^2+1)^{3/2}} \left(\frac{3\alpha^2}{\sqrt{\alpha^2+1}+\alpha}\right.$$

$$\left. + \sqrt{\alpha^2+1}\left(3\alpha\sqrt{\alpha^2+1} - 2(\alpha^2+1) - \alpha^2 + \frac{2(\alpha^2+1) - 2\alpha^2 + 3\alpha\sqrt{\alpha^2+1}}{\alpha^2+1}\right)\right)$$

Note: $(I_{3,c})$ and $(I_{3,s})$ stand for the asymptotic values of $I_{3,c}$ and $I_{3,s}$ respectively.

$$= \frac{\pi}{128\alpha(\alpha^2+1)^{3/2}} \left(\frac{3\alpha^2}{\sqrt{\alpha^2+1}+\alpha}\right.$$

$$\left. + \sqrt{\alpha^2+1}\left(\frac{3\alpha}{\sqrt{\alpha^2+1}+\alpha} - 2 + \frac{2+3\alpha\sqrt{\alpha^2+1}}{\alpha^2+1}\right)\right)$$

$$= \frac{\pi}{128\alpha(\alpha^2+1)^{3/2}} \left(3\alpha - 2\sqrt{\alpha^2+1} + \frac{2}{\sqrt{\alpha^2+1}} + 3\alpha\right)$$

$$M_3 = \frac{-3\alpha^2+1}{4(\alpha^2+1)} \left(\frac{\pi\alpha}{8(\alpha^2+1)^2}\right)$$

$$M_1+M_2+M_3 = \frac{\pi}{64}\left[\frac{1}{\alpha(\alpha^2+1)}\left(\frac{3\alpha^2-1}{(\alpha^2+1)^2} + \frac{1}{\sqrt{\alpha^2+1}\,(\sqrt{\alpha^2+1}+\alpha)}\right)\right.$$

$$\left. + \frac{1}{\alpha(\alpha^2+1)^{3/2}}\left(3\alpha + \frac{1}{\sqrt{\alpha^2+1}} - \sqrt{\alpha^2+1}\right) - \frac{2\alpha(3\alpha^2+1)}{(\alpha^2+1)^3}\right]$$

$$= \frac{\pi}{64\alpha(\alpha^2+1)^3}\left[3\alpha^2-1+(\alpha^2+1)^2-\alpha(\alpha^2+1)^{3/2}+3\alpha(\alpha^2+1)^{3/2}\right.$$

$$\left. - (\alpha^2+1)^2 - 2\alpha^2(3\alpha^2+1)\right]$$

$$= \frac{\pi}{64\alpha(\alpha^2+1)^3}\left[4\alpha^2+2\alpha(\alpha^2+1)^{3/2}-2\alpha^2(3\alpha^2+1)\right]$$

$$= \frac{\pi}{32(\alpha^2+1)^3}\left[\alpha+(\alpha^2+1)^{3/2}-3\alpha^3\right]$$

$$\rightarrow \frac{-\pi}{16\alpha^3}$$

$$I_{4,s} \sim 2\Omega \left\{ \frac{2A}{\sqrt{\alpha^2+1}} \int \frac{\xi^2(\xi^2+\alpha)(\xi^2+E)}{P^5} d\xi + \frac{A}{2(\alpha^2+1)} \int \frac{\xi^2(\xi^2+\alpha)(\alpha\xi^2-F)}{P^4} d\xi \right\}$$

$$- \frac{3\alpha^2+1}{4(\alpha^2+1)} (I_{3,s}) \text{ (See Footnote)}$$

$$= 2\Omega(N_1 + N_2 + N_3)$$

$$N_1 = \frac{2A}{\sqrt{\alpha^2+1}} \left(\frac{EI_4+3J_4}{16}\right)$$

$$= \frac{A}{8\sqrt{\alpha^2+1}} \left(\frac{E}{\alpha^2+1} \left(\frac{11}{12} I_3 - \alpha J_4\right) + 3J_4\right)$$

$$= \frac{A}{8\sqrt{\alpha^2+1}} \left(\frac{11}{12} \frac{E}{\alpha^2+1} I_3 + \left(3 - \frac{\alpha E}{\alpha^2+1}\right)\left(\frac{3}{4} J_3 - \frac{\alpha I_3}{12}\right)\right)$$

$$= \frac{A}{8\sqrt{\alpha^2+1}} \left(\left[\frac{E}{12(\alpha^2+1)} \left(11 + \frac{\alpha^2}{\alpha^2+1}\right) - \frac{\alpha}{4}\right] I_3 + \frac{3}{4} \left(3 - \frac{\alpha E}{\alpha^2+1}\right) J_3\right)$$

$$N_2 = \frac{A}{2(\alpha^2+1)} \left(\frac{3\alpha J_3 - FI_3}{12}\right)$$

$$N_3 = - \frac{3\alpha^2+1}{4(\alpha^2+1)} \frac{2A}{\sqrt{\alpha^2+1}} \left(\frac{EI_3 + 3J_3}{12}\right)$$

$$N_1 + N_2 + N_3 = \frac{A}{96(\alpha^2+1)^{5/2}} [(E(12\alpha^2+11) - 3\alpha(\alpha^2+1) - 4F(\alpha^2+1)^{3/2}$$

$$-4(3\alpha^2+1)(\alpha^2+1)E) I_3 + (9(3(\alpha^2+1)^2 - \alpha(\alpha^2+1)E) + 12\alpha(\alpha^2+1)^{3/2} - 12(3\alpha^2+1)(\alpha^2+1)) J_3]$$

$$= \frac{A}{96(\alpha^2+1)^{5/2}} [(E(-12\alpha^4-4\alpha^2+7) - 4F(\alpha^2+1)^{3/2} - 3\alpha(\alpha^2+1)) I_3$$

$$+(12\alpha(\alpha^2+1)^{3/2} + 3(\alpha^2+1)(-3\alpha^2+5) - 9\alpha(\alpha^2+1)E) J_3]$$

Note: $(I_{3,c})$ and $(I_{3,s})$ stand for the asymptotic values of $I_{3,c}$ and $I_{3,s}$ respectively.

The coefficient of I_3

$$= -2\alpha(12\alpha^4+4\alpha^2-7) + \sqrt{\alpha^2+1}\,(12\alpha^4+4\alpha^2-7)$$

$$-4\alpha(\alpha^2+1)^2 + 8\sqrt{\alpha^2+1}\,(\alpha^2+1)^2$$

$$-3\alpha(\alpha^2+1)$$

$$= -\alpha(28\alpha^4+19\alpha^2-7) + \sqrt{\alpha^2+1}\,(20\alpha^4+20\alpha^2+1)$$

The coefficient of J_3

$$= 3(3\alpha^2-5)(\alpha^2+1) + 12\alpha\sqrt{\alpha^2+1}\,(\alpha^2+1)$$

$$-18\alpha^2(\alpha^2+1) + 9\alpha\sqrt{\alpha^2+1}\,(\alpha^2+1)$$

$$= -3(\alpha^2+1)(3\alpha^2+5) + 21\alpha\sqrt{\alpha^2+1}\,(\alpha^2+1)$$

$$N_1 + N_2 + N_3 = \frac{AI_1}{1024(\alpha^2+1)^{9/2}}\,[-\alpha(28\alpha^4+19\alpha^2-7)(2\alpha^4+5\alpha^2+7)$$

$$-\alpha(\alpha^2+1)(2\alpha^2+4)(20\alpha^4+20\alpha^2+1)$$

$$+\sqrt{\alpha^2+1}((2\alpha^4+5\alpha^2+7)(20\alpha^4+20\alpha^2+1)+\alpha^2(2\alpha^2+4)(28\alpha^4+19\alpha^2-7))$$

$$+ (\alpha^2+1)[7\alpha(\alpha^2+1)^2(6\alpha^2+5)+\alpha(\alpha^2+1)(6\alpha^2+8)(3\alpha^2+5)$$

$$- \sqrt{\alpha^2+1}\,((\alpha^2+1)(3\alpha^2+5)(6\alpha^2+5)+7\alpha^2(\alpha^2+1)(6\alpha^2+8))]]$$

$$= \frac{\pi}{2048(\alpha^2+1)^5}\,[\sqrt{\alpha^2+1}\,((\alpha^2+1)[20\alpha^2(2\alpha^4+5\alpha^2+7)+2\alpha^2(28\alpha^4+19\alpha^2-7)$$

$$- (\alpha^2+1)(18\alpha^4+45\alpha^2+25)-7\alpha^2(\alpha^2+1)(6\alpha^2+8)]$$

$$- 2\alpha^4+5\alpha^2+7+2\alpha^2(28\alpha^4+19\alpha^2-7))]$$

$$+ \alpha((\alpha^2+1)[7(\alpha^2+1)^2(6\alpha^2+5)+(\alpha^2+1)(6\alpha^2+8)(3\alpha^2+5)$$

$$- (2\alpha^2+4)(20\alpha^4+20\alpha^2+1)-2\alpha^2(28\alpha^4+19\alpha^2-7)]$$

$$- (3\alpha^2+7)(28\alpha^4+19\alpha^2-7))$$

$$= \frac{\pi}{2048(\alpha^2+1)^5}\Big[\sqrt{\alpha^2+1}\ [(\alpha^2+1)^2[20(2\alpha^4+5\alpha^2+7)+2(28\alpha^4+19\alpha^2-7)$$

$$-(18\alpha^4+45\alpha^2+25)-(42\alpha^4+56\alpha^2)]$$

$$-2(28\alpha^4+19\alpha^2-7)-(20\alpha^2+19)(2\alpha^4+5\alpha^2+7)]$$

$$+\alpha[(\alpha^2+1)^2[7(\alpha^2+1)(6\alpha^2+5)+(6\alpha^2+8)(3\alpha^2+5)$$

$$-2(20\alpha^4+20\alpha^2+1)-2(28\alpha^4+19\alpha^2-7)]$$

$$-2(\alpha^2+1)(20\alpha^4+20\alpha^2+1)-(\alpha^2+5)(28\alpha^4+19\alpha^2-7)]\Big]$$

$$= \frac{\pi}{2048(\alpha^2+1)^4}(\sqrt{\alpha^2+1}\ ((\alpha^2+1)(36\alpha^4-3\alpha^2-13)-5)$$

$$-\alpha((\alpha^2+1)(-36\alpha^4+15\alpha^2+16)-136))$$

$$= \frac{\pi}{2048(\alpha^2+1)^4}[(\alpha^2+1)(\frac{36\alpha^4}{(\sqrt{\alpha^2+1}+\alpha)}-3\alpha^2\sqrt{\alpha^2+1}-15\alpha^3-13\sqrt{\alpha^2+1}-16\alpha)$$

$$-5\sqrt{\alpha^2+1}+136\alpha]$$

$$= \frac{\pi}{2048(\alpha^2+1)^4}[(\alpha^2+1)(\frac{15\alpha^3}{\sqrt{\alpha^2+1}+\alpha}(\alpha-\sqrt{\alpha^2+1})$$

$$+\frac{3\alpha^2}{\sqrt{\alpha^2+1}+\alpha}(2\alpha^2-(\alpha^2+1)-\alpha\sqrt{\alpha^2+1}))$$

$$-13\sqrt{\alpha^2+1}-16\alpha)-5\sqrt{\alpha^2+1}+136\alpha]$$

$$= \frac{\pi}{2048(\alpha^2+1)^4}[(\alpha^2+1)(\frac{-15\alpha^3}{(\alpha+\sqrt{\alpha^2+1})^2}-\frac{-6\alpha^2}{\sqrt{\alpha^2+1}+\alpha}$$

$$+\frac{3\alpha^2\sqrt{\alpha^2+1}}{(\sqrt{\alpha^2+1}+\alpha)^2}-13\sqrt{\alpha^2+1}-16\alpha)-5\sqrt{\alpha^2+1}+136\ \alpha]$$

$$\to \frac{-32\pi}{2048\alpha^5}$$

$$= \frac{-\pi}{32\alpha^5}$$

$$I_{4,s} \to \frac{-\pi\Omega}{16\alpha^5}$$

APPENDIX II - Asymptotic Values for Large α of J Integrals (see [1], p. 17).

$$J_{k,c} = \frac{\nu}{U} \int_{-\infty}^{-\ell} \psi_k \cos\left(\frac{\nu}{U}(x + \ell)\right) dx$$

let $w = \frac{x+\ell}{\ell}$

then $J_{k,c} = \Omega \int_{-\infty}^{0} \psi_k \cos(\Omega w)\, dw$

similarly

$$J_{k,s} = \Omega \int_{-\infty}^{0} \psi_k \sin(\Omega w)\, dw$$

Let

$$J_{\alpha,c} = \Omega \int_{-\infty}^{0} \psi_\alpha \cos(\Omega w)\, dw$$

$$J_{\alpha,s} = \Omega \int_{-\infty}^{0} \psi_\alpha \sin(\Omega w)\, dw$$

where

$$\psi_\alpha = \frac{c_{10} \frac{\partial\psi_1}{\partial\alpha} + c_{20} \frac{\partial\psi_2}{\partial\alpha}}{U^2 M}$$

In these terms we need ψ_α and $\left\{\psi_k\right\}_{k=1}^{4}$ for large α. To do this it is first necessary to put ξ in terms of α and w

$$4\xi^4 + 4\alpha\xi^2 - 1 + \frac{2}{w} = 0 \qquad \text{(see [1] page 44)}$$

$$2\xi^2 = -\alpha + \sqrt{\alpha^2+1 - \frac{2}{w}}$$

$$= \frac{1 - \frac{2}{w}}{\alpha + \sqrt{\alpha^2+1 - \frac{2}{w}}}$$

$$\xi = \frac{\sqrt{1 - \frac{2}{w}}}{\sqrt{2(\alpha + \sqrt{\alpha^2+1 - \frac{2}{w}})}}$$

$\psi_1)$ $$\psi_1 = \frac{\xi}{2\xi^2 + \alpha} - \frac{B}{\sqrt{\alpha^2+1}}$$

$$= \frac{\sqrt{1 - \frac{2}{w}}}{\sqrt{\alpha^2+1 - \frac{2}{w}}\ \sqrt{2(\alpha + \sqrt{\alpha^2+1 - \frac{2}{w}})}} - \frac{1}{\sqrt{\alpha^2+1}\ \sqrt{2(\alpha + \sqrt{\alpha^2+1})}}$$

$$\rightarrow \frac{\sqrt{1 - \frac{2}{w}} - 1}{2\alpha^{3/2}}$$

$\psi_2)$ $$\psi_2 = \xi - B$$

$$= \frac{\sqrt{1 - \frac{2}{w}}}{\sqrt{2(\alpha + \sqrt{\alpha^2+1 - \frac{2}{w}})}} - \frac{1}{\sqrt{2(\alpha + \sqrt{\alpha^2+1})}}$$

$$\rightarrow \frac{\sqrt{1 - \frac{2}{w}} - 1}{2\alpha^{1/2}}$$

$\psi_3)$ $$\psi_3 = w(1 - \frac{A\xi}{\sqrt{\alpha^2+1}}\ (2\xi^2 + \alpha + \sqrt{\alpha^2+1})) - \frac{\alpha^2+2+\alpha\sqrt{\alpha^2+1}}{2(\alpha^2+1)}$$

$$= w(1 - \frac{\sqrt{1 - \frac{2}{w}}}{2\sqrt{\alpha^2+1}} \sqrt{\frac{\alpha + \sqrt{\alpha^2+1}}{\alpha + \sqrt{\alpha^2+1 - \frac{2}{w}}}}\ (\sqrt{\alpha^2+1} + \sqrt{\alpha^2+1 - \frac{2}{w}}))$$

$$- \frac{\alpha^2+2+\alpha\sqrt{\alpha^2+1}}{2(\alpha^2+1)}$$

$$\rightarrow w(1 - \sqrt{1 - \frac{2}{w}}) - 1$$

$$= \frac{2}{1 + \sqrt{1 - \frac{2}{w}}} - 1$$

$$= \frac{1 - \sqrt{1 - \frac{2}{w}}}{1 + \sqrt{1 - \frac{2}{w}}}$$

ψ_4) $$\psi_4 = \frac{w^2}{2}\left(1 - \frac{A\xi}{\sqrt{\alpha^2+1}}\left(2\xi^2 + \sqrt{\alpha^2+1} + \alpha\right)\right)$$

$$-w\left(1 + \frac{A\xi}{4(\alpha^2+1)^{3/2}}\left(2(\alpha\sqrt{\alpha^2+1} - 3\alpha^2 - 1)\xi^2 - (2\alpha^3 + \sqrt{\alpha^2+1}(2\alpha^2+3)\right)\right)$$

$$- \frac{2(1-\alpha^2)\alpha\sqrt{\alpha^2+1} - (2\alpha^4 + 11\alpha^2 + 3)}{16(\alpha^2+1)^2}$$

$$\rightarrow \frac{w^2}{2}\left(1 - \sqrt{1 - \frac{2}{w}}\right) - w\left(1 - \frac{\sqrt{1 - \frac{2}{w}}}{2}\right) + \frac{1}{4}$$

$$= w\left(\frac{1}{1 + \sqrt{1 - \frac{2}{w}}} + \frac{\sqrt{1 - \frac{2}{w}}}{2} - 1\right) + \frac{1}{4}$$

$$= \frac{w}{2}\left(\frac{1 - \sqrt{1 - \frac{2}{w}}}{1 + \sqrt{1 - \frac{2}{w}}} + \sqrt{1 - \frac{2}{w}} - 1\right) + \frac{1}{4}$$

$$= \left(\frac{1}{1 + \sqrt{1 - \frac{2}{w}}}\right)^2 - \frac{1}{1 + \sqrt{1 - \frac{2}{w}}} + \frac{1}{4}$$

$$= \left(\frac{1}{1 + \sqrt{1 - \frac{2}{w}}} - \frac{1}{2}\right)^2$$

$$= \frac{1}{4}\left(\frac{1 - \sqrt{1 - \frac{2}{w}}}{1 + \sqrt{1 - \frac{2}{w}}}\right)^2$$

ψ_α) $\frac{\partial\psi_1}{\partial\alpha} \to -\frac{3}{4\alpha^{5/2}}\left(\sqrt{1-\frac{2}{w}}-1\right)$

$$\frac{\partial\psi_2}{\partial\alpha} \to -\frac{1}{4\alpha^{3/2}}\left(\sqrt{1-\frac{2}{w}}-1\right)$$

$$\frac{c_{10}}{U^2M} = \frac{(\alpha^2+1)(\alpha+\sqrt{\alpha^2+1})}{\alpha\sqrt{2(\alpha+\sqrt{\alpha^2+1})}}$$

$$\to \alpha^{3/2}$$

$$\frac{c_{20}}{U^2M} = -\frac{3\alpha^2+1+3\alpha\sqrt{\alpha^2+1})}{\alpha\sqrt{2(\alpha+\sqrt{\alpha^2+1})}}$$

$$\to -3\alpha^{1/2}$$

Therefore

$$\psi_\alpha \to \frac{o}{\alpha}$$

(Since, as will be shown, α_c and α_s are $O(\alpha)$, while the equations involving $J_{\alpha,c}$ and $J_{\alpha,s}$ are $O(1)$, $\psi_\alpha = o(\frac{1}{\alpha})$ is sufficient to show that these terms may be neglected.)

Let $\quad N_k = J_{k,c} + i\, J_{k,s}$

$$= \Omega \int_{-\infty}^{o} \psi_k\, e^{i\Omega w}\, dw$$

Let $v = -w$

$$N_k = \Omega \int_{o}^{\infty} \psi_k e^{-i\Omega v}\, dv$$

Let $w_o = \Omega \int_0^\infty (\sqrt{1 + \frac{2}{v}} - 1) e^{-i\Omega v} dv$

$$N_1 \to \frac{w_o}{2\alpha^{3/2}}$$

$$N_2 \to \frac{w_o}{2\alpha^{1/2}}$$

$$N_3 \to w_1 \equiv \Omega \int_0^\infty \left(\frac{1 - \sqrt{1 + \frac{2}{v}}}{1 + \sqrt{1 + \frac{2}{v}}}\right) e^{-i\Omega v} dv$$

$$N_4 \to w_2 \equiv \frac{\Omega}{4} \int_0^\infty \left(\frac{1 - \sqrt{1 + \frac{2}{v}}}{1 + \sqrt{1 + \frac{2}{v}}}\right)^2 e^{-i\Omega v} dv$$

Let $x = v + 1$

$$\sqrt{1 + \frac{2}{v}} = \sqrt{\frac{v+2}{v}} = \sqrt{\frac{x+1}{x-1}}$$

$$w_o = \Omega e^{-i\Omega} \int_1^\infty \left(\sqrt{\frac{x+1}{x-1}} - 1\right) e^{-i\Omega x} dx$$

$$= \Omega e^{i\Omega}(K_o(i\Omega) + K_1(i\Omega)) + i$$

$$w_1 = \Omega e^{i\Omega} \int_1^\infty \left(\frac{1 - \sqrt{\frac{x+1}{x-1}}}{1 + \sqrt{\frac{x+1}{x-1}}}\right) e^{-i\Omega x} dx$$

$$= - i e^{i\Omega} K_1(i\Omega) + (\frac{1}{\Omega} + i)$$

$$w_2 = \frac{\Omega e^{i\Omega}}{4} \int_1^\infty (x - \sqrt{x^2 - 1})^2 e^{-i\Omega x} dx$$

$$= \frac{-i}{4} (1 - \frac{4i}{\Omega} - \frac{4}{\Omega^2}) + i e^{i\Omega}(\frac{1}{2} K_o(i\Omega) - \frac{i}{\Omega} K_1(i\Omega))$$

The results for w_o, w_1, w_2 are obtained by simple maneuvers like integration by parts, see [3] page 172.

APPENDIX III - Asymptotic Values of $\phi_k(\infty)$ and $\psi_k(\infty)$

Part 1: ϕ_k

$$A = \sqrt{\frac{\sqrt{\alpha^2+1} + \alpha}{2}}$$

$$\rightarrow \sqrt{\alpha}$$

$$\phi_1 = \frac{A}{\sqrt{\alpha^2+1}}$$

$$\rightarrow \frac{1}{\sqrt{\alpha}}$$

$$\phi_2 = - A$$

$$\rightarrow - \sqrt{\alpha}$$

$$\phi_3 = \frac{\alpha + \sqrt{\alpha^2+1}}{2(\alpha^2+1)}$$

$$\rightarrow \frac{1}{\alpha}$$

$$\phi_4 = \frac{\alpha(1-2\alpha^2-2\alpha\sqrt{\alpha^2+1})}{8(\alpha^2+1)^2}$$

$$\rightarrow - \frac{1}{2\alpha}$$

$$\phi_\alpha = \frac{1}{U^2 M_o} \sum_{k=1}^{2} C_{ko} \frac{\partial \phi_k}{\partial \alpha}$$

$$= - AB\left(\frac{(\alpha^2+1)(\alpha+\sqrt{\alpha^2+1})}{\alpha} \frac{(2\alpha-\sqrt{\alpha^2+1})}{2(\alpha^2+1)^{3/2}} - \frac{(3\alpha^2+1+3\alpha\sqrt{\alpha^2+1})}{2\alpha\sqrt{\alpha^2+1}}\right)$$

$$= - \frac{1}{4\alpha\sqrt{\alpha^2+1}} \left[(\alpha+\sqrt{\alpha^2+1})(2\alpha-\sqrt{\alpha^2+1}) - (3\alpha^2+1+3\alpha\sqrt{\alpha^2+1}) \right]$$

$$= - \frac{1}{4\alpha\sqrt{\alpha^2+1}} (\alpha^2+\alpha\sqrt{\alpha^2+1} - 1 - 3\alpha^2 - 1-3\alpha\sqrt{\alpha^2+1})$$

$$= \frac{\alpha(\alpha + \sqrt{\alpha^2+1}) + 1}{2\alpha\sqrt{\alpha^2+1}}$$

$$\rightarrow 1$$

Part 2: $\tilde{\psi}_k$

$$B = \frac{1}{2A}$$

$$\rightarrow \frac{1}{2\sqrt{\alpha}}$$

$$\tilde{\psi}_1 = \frac{B}{\sqrt{\alpha^2+1}}$$

$$\rightarrow \frac{1}{2\alpha^{3/2}}$$

$$\tilde{\psi}_2 = B$$

$$\rightarrow \frac{1}{2\sqrt{\alpha}}$$

$$\tilde{\psi}_3 = \frac{\alpha^2+2+\alpha\sqrt{\alpha^2+1}}{2(\alpha^2+1)}$$

$$\rightarrow 1$$

$$\tilde{\psi}_4 = \frac{2(1-\alpha^2)\alpha\sqrt{\alpha^2+1} - (2\alpha^4+11\alpha^2+3)}{16(\alpha^2+1)^2}$$

$$\rightarrow -\frac{1}{4}$$

$$\tilde{\psi}_\alpha = \frac{1}{U^2 M_o} \sum_{k=1}^{2} C_{ko} \frac{\partial \tilde{\psi}_k}{\partial \alpha}$$

$$= -\frac{B^2}{2\alpha\sqrt{\alpha^2+1}} \left((\alpha^2+1)(\alpha+\sqrt{\alpha^2+1})\left(\frac{\sqrt{\alpha^2+1}+2\alpha}{\alpha^2+1}\right)\right.$$

$$\left. - (3\alpha^2+1+3\alpha\sqrt{\alpha^2+1})\right)$$

$$= 0$$

APPENDIX IV - Asymptotic Form of Pressure, Lift, and Moment Terms

A) Pressure

Along the wetted surface of the hydrofoil, the pressure terms are functions of η alone, where η is given by

$$\eta^{\pm} = -\sqrt{\alpha \mp \sqrt{\frac{\ell - x}{\ell + x}}} ,$$

where η^{+} is the value on the upper surface and η^{-} the value on the lower surface. Let $\Delta = \eta^{+} - \eta^{-}$

then

$$\Delta = \sqrt{\alpha + \sqrt{\frac{\ell - x}{\ell + x}}} - \sqrt{\alpha - \sqrt{\frac{\ell - x}{\ell + x}}}$$

For large values of α

$$\Delta = \sqrt{\alpha}\left(\sqrt{1 + \frac{1}{\alpha}\sqrt{\frac{\ell - x}{\ell + x}}} - \sqrt{1 - \frac{1}{\alpha}\sqrt{\frac{\ell - x}{\ell + x}}}\right)$$

$$\sim \sqrt{\alpha}\left(\frac{1}{\alpha}\sqrt{\frac{\ell - x}{\ell + x}}\right)$$

$$= \frac{1}{\sqrt{\alpha}}\sqrt{\frac{\ell - x}{\ell + x}}$$

Let $\delta_k = \phi_k^{+} - \phi_k^{-}$, where ϕ_k^{+} and ϕ_k^{-} are the upper and lower surface values respectively.

Let $\delta_\alpha = \frac{1}{U^2 M} \sum_{1}^{2} C_{ko} \frac{\partial}{\partial \alpha} (\phi_k^{+} - \phi_k^{-})$

$$\delta_1 = -\frac{1}{\eta^+} + \frac{1}{\eta^-}$$

$$= \frac{\Delta}{\eta^+\eta^-}$$

$$\sim \frac{1}{\alpha^{3/2}}\sqrt{\frac{\ell-x}{\ell+x}}$$

$$\delta_2 = \Delta$$

$$\sim \frac{1}{\alpha^{1/2}}\sqrt{\frac{\ell-x}{\ell+x}}$$

Observe that $(\eta^{\pm 2} + \alpha)^2 + 1 = \frac{2\ell}{\ell+x}$ (for both signs)

$$\delta_3 = \frac{-2A}{\sqrt{\alpha^2+1}}\left(\frac{\ell+x}{2\ell}\right)\left(\eta^{+3} - \eta^{-3} + (\sqrt{\alpha^2+1} - 2\alpha)(\eta^+ - \eta^-)\right)$$

$$\eta^{+3} - \eta^{-3} = \Delta(\eta^{+2} + \eta^+\eta^- + \eta^{+2})$$

$$\sim 3\alpha\Delta$$

Therefore

$$\delta_3 \sim -\frac{A}{\alpha}\left(\frac{\ell+x}{\ell}\right)(2\alpha\Delta)$$

Since $A \sim \sqrt{\alpha}$

$$\delta_3 \sim -2\left(\frac{\ell+x}{\ell}\right)\sqrt{\frac{\ell-x}{\ell+x}}$$

$$= -\frac{2\sqrt{\ell^2-x^2}}{\ell}$$

$$\delta_4 = -\frac{2A}{\sqrt{\alpha^2+1}}\left(\frac{\ell+x}{2\ell}\right)^2\left(\eta^{+3}-\eta^{-3}+(\sqrt{\alpha^2+1}-2\alpha)(\eta^+-\eta^-)\right)$$

$$-\frac{A}{2(\alpha^2+1)}\left(\frac{\ell+x}{2\ell}\right)(\alpha(\eta^{+3}-\eta^{-3})+[\alpha\sqrt{\alpha^2+1}-2(\alpha^2+1)](\eta^+-\eta^-)-\frac{3\alpha^2+1}{4(\alpha^2+1)}\delta_3$$

$$\sim -\frac{2A}{\alpha}\left(\frac{\ell+x}{2\ell}\right)^2(2\alpha\Delta)-\frac{A}{2\alpha^2}\left(\frac{\ell+x}{2\ell}\right)(2\alpha^2\Delta)-\frac{3}{4}\delta_3$$

$$\sim \sqrt{\frac{\ell-x}{\ell+x}}\left[-\left(\frac{\ell+x}{\ell}\right)^2-\frac{\ell+x}{2\ell}+\frac{3}{2}\left(\frac{\ell+x}{\ell}\right)\right]$$

$$\sim \sqrt{\frac{\ell-x}{\ell+x}}\left(\frac{\ell+x}{\ell}-\left(\frac{\ell+x}{\ell}\right)^2\right)$$

$$= -\frac{x\sqrt{\ell^2-x^2}}{\ell^2}$$

$$\delta_\alpha \sim \frac{B}{2\alpha}(\alpha^2+1)(\alpha+\sqrt{\alpha^2+1})\frac{\partial}{\partial\alpha}\delta_1-\frac{B}{2\alpha}(3\alpha^2+1+3\alpha\sqrt{\alpha^2+1})\frac{\partial}{\partial\alpha}\delta_2$$

$$\sim B\alpha^2\left(-\frac{3}{2\alpha^{5/2}}\sqrt{\frac{\ell-x}{\ell+x}}\right)-3B\alpha\left(-\frac{1}{2\alpha^{3/2}}\sqrt{\frac{\ell-x}{\ell+x}}\right)$$

Since $B \to \frac{1}{2\sqrt{\alpha}}$

$$\delta_\alpha \sim -\frac{3}{2\alpha}\sqrt{\frac{\ell-x}{\ell+x}}+\frac{3}{2\alpha}\sqrt{\frac{\ell-x}{\ell+x}}$$

$$= o\left(\frac{1}{\alpha}\right)$$

B) Lift

$$I_1(-\alpha) = \frac{\pi A}{\sqrt{\alpha^2+1}}$$

$$\sim \frac{\pi}{\alpha^{1/2}}$$

$$R_1 = \frac{2\ell(\sqrt{\alpha^2+1} - 2\alpha)}{\alpha^2+1} \quad I_1$$

$$\sim - \frac{2\pi\ell}{\alpha^{3/2}}$$

$$R_2 = - 2\ell I_1$$

$$\sim - \frac{2\pi\ell}{\alpha^{1/2}}$$

$$R_3 = \frac{\pi\ell}{2(\alpha^2+1)^2} \left(2\alpha^4+3\alpha^2+3+2\alpha(\alpha^2+1)^{3/2}\right)$$

$$\sim 2\pi\ell$$

$$R_4 = \frac{\pi\ell}{4(\alpha^2+1)^3} \left(\alpha(\alpha^2+1)(\sqrt{\alpha^2+1} + \alpha) - 4\alpha^2\right)$$

$$\sim \frac{\pi\ell}{2\alpha^2}$$

(Note:) $$R_\alpha = \frac{-\pi\ell}{\alpha(\alpha^2+1)^{3/2}} \left(4\alpha + \sqrt{\alpha^2+1}\right)$$

$$\sim - 5 \frac{\pi\ell}{\alpha^3}$$

C) Moment

$$S_1 = - \frac{\ell^2(\alpha(\alpha^2+13) + \sqrt{\alpha^2+1}\ (\alpha^2-5))I_1}{2(\alpha^2+1)^2}$$

$$\sim - \frac{\pi\ell^2}{\alpha^{3/2}}$$

$$S_2 = - \frac{\ell^2(3+\alpha^2+\alpha\sqrt{\alpha^2+1})I_1}{2(\alpha^2+1)}$$

$$\sim - \frac{\pi\ell^2}{\alpha^{1/2}}$$

$$S_3 = \frac{\pi\ell^2}{4(\alpha^2+1)^3}\ (4\alpha^6+o(\alpha^6)+\alpha\sqrt{\alpha^2+1}\ (4\alpha^4+o(\alpha^4))$$

$$\sim 2\pi\ell^2$$

$$S_4 = \frac{\ell^2}{64(\alpha^2+1)^4}\ (8\alpha^8+o(\alpha^8) + 4\alpha\ \alpha^2+1\ (2\alpha^6+o(\alpha^6)))$$

$$\sim \frac{\pi\ell^2}{4}$$

(Note) $$S_\alpha = \frac{-\pi\ell^2}{2\alpha(\alpha^2+1)^{5/2}}\ (\alpha(-5\alpha^2+19) + \sqrt{\alpha^2+1}\ (-5\alpha^2+4))$$

$$\sim \frac{5\pi\ell^2}{\alpha^3}$$

Note: In (1), R_α and S_α defined slightly differently (i.e. the factor U^2M was included).

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